MATH3210 - SPRING 2024 - SECTION 004

EXAM 2 - REVIEW SHEET

How to use this review sheet

On the exam or problems, you may use any of the definitions and theorems stated on the review sheet, *unless you are explicitly asked to prove a theorem listed here.* Any unnamed theorem you may use without citing. If you use a named theorem, cite that theorem by name when invoking its conclusions. Please don't hesitate to ask questions. I've proofread this, but typos may still lurk!

1. Continuity

Definition 1. If $D \subset \mathbb{R}$ is a domain (usually an interval, ray, or all of $\mathbb{R}$), and $f : D \to \mathbb{R}$ is a function, then:

- f is *continuous at* $x \in D$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $y \in D$ such that $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$,
- f is *(pointwise) continuous* (or just *continuous*) if and only if it is continuous at every $x \in D$, and
- f is *uniformly continuous* if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $x, y \in D$ such that $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$.

Theorem 1 (Extreme Value Theorem)**.** *If $I = [a, b]$ is closed and bounded, and $f : I \to \mathbb{R}$ is continuous, then f is bounded on I, and f achieves its minimum and maximum values (ie, there exists $c \in I$ such that $f(c) = \sup_I f(x)$, and similarly for the minimum value).*

Theorem 2 (Intermediate Value Theorem)**.** *If $I = [a, b]$ is closed and bounded, $f : I \to \mathbb{R}$ is continuous, and y lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.*

Theorem 3. *If $I = [a, b]$ and $f : I \to \mathbb{R}$ is strictly increasing and continuous, then $f(I) = [c, d]$ is a closed interval and there exists a continuous inverse $g : [c, d] \to \mathbb{R}$.*

Theorem 4. *If $I = [a, b]$ is a closed interval, any continuous function on I is uniformly continuous.*

Theorem 5. *If $f : D \to \mathbb{R}$ is a uniformly continuous function on a domain D, and $\{x_n\}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is a Cauchy sequence.*

Definition 2. Let I be an interval with endpoints a and b, each of which may or may not belong to I. Then $\bar{I}$, the *closure of* I, is the closed interval $[a, b]$. If $f : I \to \mathbb{R}$ is a function, an *extension of f to $\bar{I}$* is a function $F : \bar{I} \to \mathbb{R}$ such that for all $x \in I$, $F(x) = f(x)$.

Theorem 6. *If f is a continuous function on an interval I, then f is uniformly continuous on I if and only if it has a continuous extension to $\bar{I}$.*

2. Limits

Definition 3. Let I be an interval, ray or the real line and $a \in I$ be an interior point.

- We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
- If a is an interior point, right-hand endpoint, or ∞, we say that $\lim_{x \to a^-} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists some $m < a$ such that if $m < x < a$, $|f(x) - L| < \varepsilon$.
- If a is an interior point, left-hand endpoint, or $-\infty$, we say that $\lim_{x \to a^+} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists some $m > a$ such that if $a < x < m$, $|f(x) - L| < \varepsilon$.

Theorem 7. *Let I be an interval and $a \in I$ be an interior point. Then*

$$\lim_{x \to a} f(x) = L \textit{ if and only if } \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Theorem 8 (Limit Arithmetic Theorem)**.** *Assume all limits involved exist as real numbers.*[1]

(a) *If $f(x)$ is the constant function with value c, then* $\lim_{x \to a} f(x) = c$

(b) $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

(c) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(d) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(e) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, *assuming that* $\lim_{x \to a} g(x)$ *exists.*

3. Derivatives

Definition 4. If $f : I \to \mathbb{R}$ is a function defined on an interval, and $a \in I$ is an interior point, we say that f is *differentiable at a* if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We denote the limit by $f'(a)$. We say that f is *differentiable on I* if it is differentiable at every interior point of I.

Theorem 9. *If f is differentiable at a, then it is continuous at a.*

Theorem 10 (Derivative arithmetic theorem)**.** *Assume that f and g are functions which are differentiable at a. Then*

(a) $(cf)'(a) = cf'(a)$

(b) $(f + g)'(a) = f'(a) + g'(a)$

(c) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

(d) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$, *assuming that* $g(a) \neq 0$

Theorem 11 (The Chain Rule)**.** *If g is differentiable at a, and f is differentiable at $b = g(a)$, then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Theorem 12. *If $f : I \to \mathbb{R}$ is differentiable and $f'(x) > 0$ on I, then the inverse function g of f (provided by Theorem 3) is differentiable, and*

$$g'(y) = \frac{1}{f'(g(y))}.$$

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[1] Note that this is a nontrivial assumption! You need all limits to exist to apply this thoerem.

Theorem 13. *If f is a continuous function on a closed bounded interval, f achieves its maximum value at interior point c, and f is differentiable at c, then $f'(c) = 0$.*

3.1. **The mean value theorem and corollaries.**

Theorem 14 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on its interior points, then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 15. *If f is a continuous function defined on an interval I, f is constant if and only if $f'(x) = 0$ for every interior point $x \in I$.*

Corollary 16. *If f and g are differentiable functions on an interval (a, b), and $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists some $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

Theorem 17. *Let f be differentiable on an open interval, ray or all of $\mathbb{R}$. If $f'(x)$ is bounded above, then f is uniformly continuous.*

3.2. **L'Hopital's rule.**

Theorem 18 (Cauchy Mean Value Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable at interior points. If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem 19 (L'Hopital's rule, interior point version)**.** *Let f and g be differentiable functions on an interval I, and $a \in I$ be such that $f(a) = g(a) = 0$, but $g(x) \neq 0$ for any other $x \in I$. Then if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists,*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Theorem 20 (L'Hopital's rule, right-hand version)**.** *Let f and g be differentiable functions on an open interval, ray, or all of $\mathbb{R}$. Let b denote the right hand endpoint of the interval, or ∞ in the case of a right-facing ray or $\mathbb{R}$. Assume that $g(x)$ and $g'(x)$ are both nonzero at every point of its domain, but either*

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^-} g(x) = 0 \qquad \textit{or} \qquad \lim_{x \to b^-} f(x) = \lim_{x \to b^-} g(x) = \infty.$$

Then when $\lim_{x \to b^-} \frac{f'(x)}{g'(x)}$ exists,

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{x \to b^-} \frac{f'(x)}{g'(x)}.$$